Math 444 - Homework 12 **Name:** ______________________

1. Use power series to find the orders of the following zeros for the indicated functions.

   (a) $f(z) = 1 + \cos(z)$ at $z_0 = \pi$.

   (b) $g(z) = z^3 \sin(z^2)$ at $z_0 = 0$.

2. Find all isolated singularities for the following functions and classify them as removable, poles, or essential. If the singularity is a pole, find its order.

   (a) $f(z) = \dfrac{z}{e^z - 1}$.

   (b) $g(z) = \dfrac{1}{(z^2+1)^3(z-1)^2}$.

3. Let D be an open simply connected domain and suppose that $f : D \to \mathbb{C}$ is holomorphic. Use the open mapping principle to prove that if $|f(z)| = 1$ for all $z \in D$, then f is a constant.

4. Let C be a simple, closed, piecewise smooth curve in an open simply connected domain D. Suppose that f is holomorphic on D, and $|f(z)| = 1$ for all $z \in C$. Prove that f contains a zero inside C or f is constant on D. Hint: if f has no zero inside C, then what does the maximum modulus principle say about f and $1/f$?

5. Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. Find the winding numbers of $f \circ \gamma$ and $g \circ \gamma$ around the origin for:

(a) $f(z) = \dfrac{z}{e^z - 1}$

(b) $g(z) = \dfrac{\sin^2 z}{z^5(z^2+1)}$

6. Find a Laurent series for $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ in each of the following domains.

(a) $1 < |z| < 2$

(b) $|z| > 2$

7. How many roots does the polynomial $z^5 + 5z^2 + 1$ have in the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$? Explain your answer using Rouche's theorem.