Homework 8 - Math 243 Name: ______________________

Each of the following linear systems has a single parameter a. Calculate the trace and determinant for each system, and use them to find the values of a where the type of equilibrium changes.

1. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} a & -1 \\ 2 & 0 \end{bmatrix} \mathbf{x}.$

2. $\dfrac{d\mathbf{y}}{dt} = \begin{bmatrix} a & a \\ 1 & 0 \end{bmatrix} \mathbf{y}.$

3. Consider the 2-parameter family $\dfrac{d\mathbf{y}}{dt} = \begin{bmatrix} 1 & a \\ -b & 0 \end{bmatrix} \mathbf{y}$. Describe in words how the type of equilibrium depends on the parameters a and b.

4. If a 2-by-2 matrix A has one eigenvalue equal to 4, express the trace and determinant of A as functions of the other eigenvalue λ. What types of equilibria are possible for the system $\mathbf{x}' = A\mathbf{x}$ with different values of λ?

Use the matrix exponential function (on a computer) to solve the following initial value problems.

5. $$\begin{aligned} x' &= x + 2y \\ y' &= 3x - 4y \\ x(0) &= 1 \\ y(0) &= 0. \end{aligned}$$

6. $$\begin{aligned} x' &= 2x + y \\ y' &= 2y \\ x(0) &= 1 \\ y(0) &= 2. \end{aligned}$$

7. $$\begin{aligned} x' &= 3x - y \\ y' &= -x + 3y \\ z' &= 5x - 5y - 6z \\ x(0) &= 4 \\ y(0) &= 2 \\ z(0) &= -3. \end{aligned}$$

8. $$\begin{aligned} x' &= -x + 6y \\ y' &= -6x - y \\ x(0) &= 3 \\ y(0) &= 0. \end{aligned}$$

9. Use the definition of the matrix exponential $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$ to calculate the matrix exponential e^{tB} when $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hint: Calculate $(tB)^2$. What do you notice?

10. The matrix $A = \begin{bmatrix} 2 & -3 \\ 0 & -1 \end{bmatrix}$ has eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$.

(a) Use the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ to find the inverse of $V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) Calculate $D = V^{-1}AV$ by hand and show that it is a diagonal matrix.

(c) Calculate $e^{tA} = Ve^{tD}V^{-1}$ by hand. Recall that for a diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $e^{tD} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$.