Formula Sheet

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrating Factors

The general solution of $y' + f(t)y = g(t)$ is

$$y(t) = \frac{\int e^{F(t)} g(t)\, dt}{e^{F(t)}}$$

where $F(t)$ is any antiderivative of $f(t)$.

2-by-2 Matrix Characteristic Polynomial

$$\lambda^2 - \lambda \operatorname{tr} A + \det A.$$

Hamiltonian Systems

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

Linear Systems

For a linear system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$:

Straight-line solutions $\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$	**Matrix exponential solution** $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$
Complex eigenvalues If $\lambda = \alpha \pm i\beta$ is a complex eigenvalue with eigenvector $\mathbf{v}$, then the real and imaginary parts of $e^{\alpha t}(\cos(\beta t) \pm i \sin(\beta t))\mathbf{v}$ are both real-valued solutions.	**Repeated eigenvalues** If A is a 2-by-2 matrix with a repeated eigenvalue λ, then the solution is $\mathbf{x}(t) = e^{\lambda t}(I + t(A - \lambda I))\mathbf{x}(0).$

Second Order Linear Equations

For a homogeneous differential equation $y'' + by' + cy = 0$ with real coefficients:

	Distinct real roots	**Complex roots**	**Repeated real roots**
Roots of $\lambda^2 + b\lambda + c$	λ_1, λ_2	$\lambda = \alpha \pm i\beta$	λ
General solution	$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$	$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$	$y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

Guessing a Particular Solution (Method of Undetermined Coefficients)

Forcing term	**Good guess**	**Next option**
$at + b$	$y_p = At + B$	
e^{kt}	$y_p = Ae^{kt}$	Multiply last guess by t
$\cos \omega t$ or $\sin \omega t$	$A \cos \omega t + B \sin \omega t$	Complexify

Laplace transforms

Original function	Laplace transform	Comments
e^{at}	$\frac{1}{s-a}$	$s > a$
$e^{at}t^n$, $n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$H(t-c)$	$\frac{e^{-cs}}{s}$	$-\infty < s < \infty$
$\delta(t-c)$	e^{-cs}	$-\infty < s < \infty$
$\frac{d}{dt}f(t)$	$sF(s) - f(0)$	First derivative rule
$\frac{d^2}{dt^2}f(t)$	$s^2F(s) - sf(0) - f'(0)$	Second derivative rule
$e^{at}f(t)$	$F(s-a)$	First exponential shift rule
$H(t-c)f(t-c)$	$e^{-cs}F(s)$	Second exponential shift rule