Final Exam Review - Math 243

1. Consider the linear system below.

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = x$$

   (a) Re-write the system in matrix form.

   (b) The eigenvectors for the system above are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue $\lambda_1 = -1$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_2 = 2$. Use these to find the general solution for the linear system.

   (c) Find the solution to the initial value problem $x(0) = 1$ and $y(0) = 2$.

2. Consider the forced harmonic oscillator

$$y'' + 5y' + 6y = 6t - 1$$

   (a) Find the general solution.

   (b) If the forcing term were removed, would this oscillator be over-damped, under-damped, or critically damped?

3. Consider the following one-parameter family of linear systems

$$\frac{dY}{dt} = AY \quad \text{where} \quad A = \begin{bmatrix} 2 & -a \\ 1 & 0 \end{bmatrix}.$$

(a) Sketch the curve determined by the parameter a in the trace-determinant plane.

(b) Describe the different types of equilibrium behaviors exhibited by the system as a varies.

4. Consider the system below.

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 2x \end{aligned}$$

(a) Verify that the system is a Hamiltonian system.

(b) Find the Hamiltonian function H.

(c) Use the Hamiltonian to sketch some solution curves for this system (including arrows to indicate direction).

5. A patient is being treated with morphine for their pain. The amount of morphine $y(t)$ (measured in micrograms per liter) in the patient's bloodstream after t hours can be modeled by the differential equation
$$y' + 0.1\, y = 30\, \delta(t-4).$$

(a) Suppose the patient initially has 100 μg/L of morphine in their blood. In the model above, the amount of morphine in the patient would decay exponentially if it weren't for an additional injection of morphine that happens at time $t = 4$ hours. How much extra morphine is the patient given at that time?

(b) Apply the Laplace transform to the differential equation above and solve for $Y(s)$.

(c) Solve the equation by computing the inverse Laplace transform of $Y(s)$.

6. Consider the population model

$$\frac{dP}{dt} = -\tfrac{1}{25}P^2 + 4P - C$$

for a species of fish in a lake (here dP/dt is measured in fish/year). Here C is a parameter that represents the number of fish caught per year.

(a) Draw a bifurcation diagram showing how the equilibrium fish populations depend on C. Include the phase lines at three different values of C (I recommend $C = 75$, 100, and something bigger than 100).

(b) What is the maximum catch C that would still allow the fish to have a chance of survival in the lake?

(c) Given the possibility of unexpected perturbations of the population not included in the model, what do you think would happen to the actual fish population if the yearly catch determined in part (b) was actually taken every year?

7. A 1000 L tank is filled with salt water. The water currently contains 20 kg of dissolved salt. Suppose that fresh water is added to the tank at a rate of 10 L per minute and an equal amount of water drains out of the bottom so that the water level stays constant. Assume that the water draining from the bottom is well mixed.

   (a) Find a differential equation that models the amount of salt in the tank.

   (b) Solve the differential equation to find an equation for the amount of salt in the tank as a function of time.

8. Consider the differential equation $y' = 2y + \cos t$.

   (a) Find the general solution.

   (b) Solve the initial value problem $y(0) = 0$.

9. Consider the differential equation $\dfrac{dy}{dt} = 3y^{2/3}$.

(a) Verify that $y(t) = 0$ is a solution to the differential equation.

(b) Verify that $y(t) = t^3$ is also a solution to the differential equation.

(c) One of the most important theorems about differential equations is the following:

Uniqueness Theorem. *If* $\dfrac{dy}{dt} = f(t, y)$ *is a differential equation where both* f *and its partial derivative* f_y *are continuous for all points* (t, y) *near* (t_0, y_0)*, then there exists a unique solution* $y(t)$ *defined on an open interval around* t_0 *such that* $y(t_0) = y_0$*.*

Given that solutions of a differential equation are supposed to be unique, how is it possible that the differential equation above has two different solutions that pass through the point $(0, 0)$? Explain why this does not contradict the Uniqueness Theorem.

10. Consider the following nonlinear system of equations.

$$\frac{dx}{dt} = x^2 - y$$
$$\frac{dy}{dt} = 2y^2 - 2$$

(a) Identify all of the equilibrium points for this system.

(b) Find the Jacobian matrix at each equilibrium and use it to classify the type for each (sink, source, spiral sink, spiral source, or saddle).

11. Consider the linear system

$$\frac{dY}{dt} = \begin{bmatrix} -4 & 3 \\ -3 & -4 \end{bmatrix} Y.$$

(a) Verify that $\begin{bmatrix} 1 \\ i \end{bmatrix}$ as an eigenvector of $\begin{bmatrix} -4 & 3 \\ -3 & -4 \end{bmatrix}$. What is its corresponding eigenvalue?

(b) Find the general (real-valued) solution of the system above.

12. Consider the initial value problem $y' = y^2 - 2y + 1$ with $y(0) = 0$.

(a) Draw a rough sketch of the slope field for this differential equation.

(b) Use Euler's method with $\Delta t = \frac{1}{2}$ to approximate the solution over the interval $0 \leq t \leq 1$.

(c) Use separation of variables to solve the initial value problem.