Midterm 2 Review **Math & Society**

The following problems are similar to ones you might see on the midterm exam. There is a also a list of terms & facts you should memorize on the last page.

1. A school district has 3 different elementary schools. The enrollment (number of students) for each of the 3 schools is shown below. The school district wants to apportion its elementary school teachers to the schools proportional to the number of students.

School	A	B	C	Total
Enrollment	69	570	641	1280

(a) If the district has 120 elementary school teachers, what is the standard divisor? What are its units?

Solution: The standard divisor is 10.67 students per teacher.

(b) What is the standard quota for each school?

Solution: The standard quotas are $A : 6.46875$, $B : 53.4375$, and $C : 60.0938$.

(c) If the district uses Hamilton's method to apportion the teachers, how many teachers will each school get?

Solution: A gets 7, B gets 53 and C gets 60.

(d) What if the district hires one more teacher, increasing the number of elementary school teachers to 121. What would Hamilton's method apportionment be in that case?

Solution: A gets 6, B gets 54 and C gets 61.

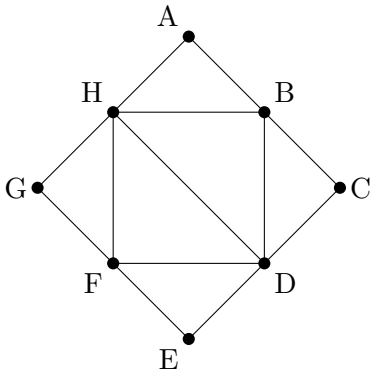
(e) Which of the following apportionment problem best describes the situation above?

A. Alabama paradox
B. New states paradox
C. Population paradox
D. Lower quota violation
E. Upper quota violation

(f) If the district used Jefferson's method to apportion the teachers, how should it adjust the standard divisor to get a divisor that apportions the teachers correctly?

A. Increase the divisor.
B. Decrease the divisor.
C. Don't change the divisor.
D. There is not enough information to know which way to adjust the divisor.

2. Consider the graph below.



(a) Does the graph have an Euler circuit? Why or why not?

Solution: No, not all of the vertices have even degrees.

(b) Does the graph have an Euler path? Why or why not?

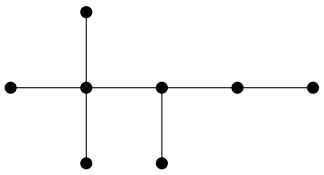
Solution: Yes, there are exactly two odd degree vertices (H and D).

(c) What is the sum of the degrees of the vertices in this graph?

Solution: A, C, E, and G have degree 2. B and F have degree 4, and H, D both have degree 5, so the total is 26.

3. Draw a tree that has a degree 4 vertex, a degree 3 vertex, and a degree 2 vertex.

Solution: There are many possible answers, but here is one:



4. Use the mileage chart below to find the minimum spanning tree connecting these five cities.

	Boston	Buffalo	Chicago	Columbus	Louisville
Boston	–	446	963	735	941
Buffalo	446	–	522	326	532
Chicago	963	522	–	308	292
Columbus	735	326	308	–	209
Louisville	941	532	292	209	–

Solution: Start with Columbus to Louisville (209). Then add Chicago to Louisville (292). You can't add Chicago to Columbus (308), so the next cheapest route is Buffalo to Columbus (326). Then add Buffalo to Boston (446). Now all five cities are connected, so you are done.

5. Which of the following graphs must be a tree? Circle all that are.

   A. An ancestry graph. People are vertices; edges connect parents to their children.

   B. Friends network. People are vertices; edges connect people who are friends.

   C. Railway network. Cities are vertices; railway routes are the edges.

6. If a graph has 100 vertices, and every vertex is degree 3, then how many edges does the graph have?

Solution: Since the sum of the degrees of the vertices is twice the number of edges, there must be 150 edges.

7. Some people argue that the United States should switch back to Hamilton's method to apportion the seats in Congress. Which of the following is *not* an advantage of Hamilton's method.

   A. There are no paradoxes.

   B. No state ever gets more than its upper quota.

   C. No state ever gets less than its lower quota.

8. Let $v = \begin{pmatrix} 0.2 & 0.2 & 0.6 \end{pmatrix}$ and let $Q = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(a) Calculate vQ. Based on your calculation, is v a stationary distribution for Q?

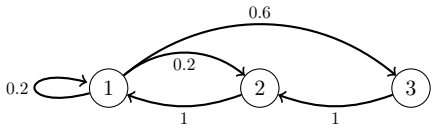
Solution:

$$vQ = \begin{pmatrix} 0.24 & 0.64 & 0.12 \end{pmatrix}.$$

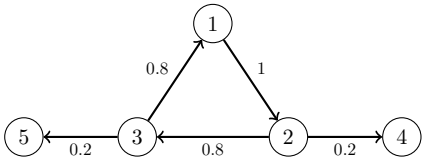
Since vQ is not the same as v, it is not a stationary distribution.

(b) The matrix $Q = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is a transition matrix for a Markov chain. Draw and label the graph for that Markov chain.

Solution:



9. Consider the Markov chain shown below.



(a) What are the strongly connected components of this Markov chain? Which components are final?

Solution: There are 3 strongly connected components. They are $\{1, 2, 3\}$, $\{4\}$, and $\{5\}$. The last two are final.

(b) The transition matrix for this Markov chain is $Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0.8 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. If you raise this matrix to a large power k (anything greater than 100), you will get

$$Q^k = \begin{pmatrix} 0 & 0 & 0 & 0.555 & 0.444 \\ 0 & 0 & 0 & 0.555 & 0.444 \\ 0 & 0 & 0 & 0.444 & 0.555 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If you start in state 3, what is the probability that you will eventually end up in state 4?

Solution: The probability is the number in row 3, column 4 : 0.444.

10. About 40% of American adults are obese. About 10% of American adults have type II diabetes. And about 9% of American adults are both obese and have type II diabetes. Let A be the event that a randomly selected American adult is obese, and let B be the event that they have type II diabetes.

(a) Are A and B independent? Explain how you can tell.

Solution: A and B are not independent because $P(A \text{ and } B) = 9\%$ which is not $P(A) \cdot P(B) = (0.10)(0.40) = 4\%$.

(b) Find $P(\text{not } A)$.

Solution:

$$P(\text{not } A) = 100\% - 40\% = 60\%.$$

(c) Find $P(A \text{ or } B)$.

Solution:

$$P(A \text{ or } B) = 40\% + 10\% - 9\% = 41\%.$$

(d) Find $P(A \mid B)$.

Solution:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = 9/10 = 90\%.$$

Midterm 2 Study Guide

Midterm 2 will focus on the following topics. Make sure you know all of the terms listed in bold.

Apportionment Know how each of the following apportionment methods work: **Hamilton's**, **Jefferson's**, **Adam's**, and **Webster's methods**. Be able to identify the **standard divisor** and **standard quotas**. Know the advantages and disadvantages of the different methods:

- Hamilton's method has no **quota violations**, but it can have the **Alabama paradox**, the **new states paradox**, and the **population paradox**.
- The divisor methods (Adams, Jeffersons, and Websters) don't have paradoxes, but they can all have quota violations. Jefferson's method is biased in favor of big states, so they sometimes get too many seats (**upper quota violations**), Adam's method is biased against big states (so they can have **lower quota violations**). Webster's method (and also the Huntington-Hill method that Congress currently uses) is not biased, but it can have both upper and lower quota violations.

Graph Theory Know what the **degree** of a vertex is. Know that the sum of the degrees of all the vertices in a graph is twice the number of edges. Know that a **tree** is a connected graph with no cycles. For a tree you should know that $V = E + 1$ and what that means. You should know when a graph has an **Euler path** or an **Euler circuit** and what those are.

Markov chains Be able to convert a description of a Markov chain to a graph and/or a **transition matrix**. Know how to multiply matrices, and understand the meaning of a power of a transition matrix. You should also know how to find the **strongly connected components** of a Markov chains and be able to recognize the **final components**. If the powers Q^k of the transition matrix Q converge, then the rows of the matrix Q^k when k is large are each **stationary distributions** v of the Markov chain which means that

$$vQ = v.$$

Probability You should be able to determine if all of the elements of a **sample space** are **equiprobable** and how to calculate the probability of an **event** in an equiprobable model. Know the basic probability rules: **Complementary events**, **Addition rule**, and **Multiplication rule for independent events**. Know that two events are **independent** when knowing that one has happened does not change the probability for the other. Know how to calculate a **weighted average**, and how to use weighted averages to find the **expected value** of a probability model with numerical outcomes. Finally, you should know the **Law of Large Numbers**.