Calculus I - Math 141 **Final Exam**

Name: ___________________________

You must show all work to earn full credit. No calculators allowed. If you do not have room in the given space to answer a question, use the back of another page and indicate clearly which work goes with which problem.

Problem	Maximum Points	Your Score
1	8	
2	9	
3	8	
4	9	
5	8	
6	6	
7	8	
8	8	
9	6	
10	6	
11	6	
12	4	
13	8	
14	6	
Total:	100	

1. (8 points) Let $f(x) = x^3 - 3x^2 + 4$.

(a) Find the intervals of increase/decrease for $f(x)$.

Solution:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

So the critical points are $x = 0$ and $x = 2$. Testing the value of $f'(x)$ at points to the left, right, and middle of these critical points, we find the intervals of increase $(-\infty, 0)$ and $(2, \infty)$ and the interval of decrease $(0, 2)$.

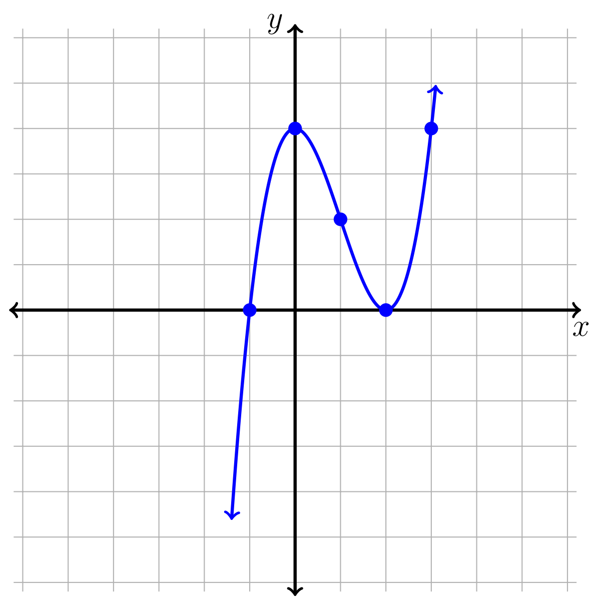
(b) Find the intervals of concavity.

Solution:

$$f''(x) = 6x - 6$$

which has critical point $x = 1$. Then $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) Sketch a graph of the function using the axis provided below.



Solution: Plot the y-values at the critical points $x = 0, 2$, and the inflection point $x = 1$, and also at some other easy points like $x = 3$ and $x = -1$. Then connect the dots, making sure it is increasing and decreasing when it should be, and concave the right way.

2. (9 points) Find the following derivatives.

(a) $\dfrac{d}{dx}\dfrac{x^2+\sqrt{x}}{x}$

Solution: Break up the fraction first: $\dfrac{d}{dx}\left(\dfrac{x^2+\sqrt{x}}{x}\right) = \dfrac{d}{dx}\left(x+x^{-1/2}\right) = 1-\frac{1}{2}x^{-3/2}$

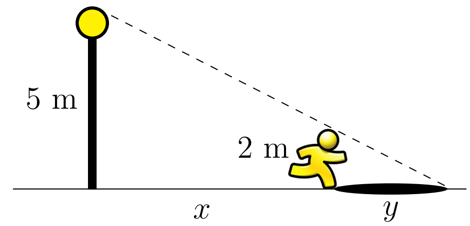
(b) $\dfrac{d}{d\theta}\cos(\theta^2+2)$

Solution: Chain rule: $-2\theta\sin(\theta^2+2)$.

(c) $\dfrac{d}{dx}x^2\sin 2x$

Solution: Product rule (with a little chain rule too): $2x\sin 2x + 2x^2\cos 2x$.

3. (8 points) A jogger, who is 2 meters tall, passes under a 5 meter street lamp running at 3 m/s.



(a) Use similar triangles to find a formula relating x and y.

Solution:

$$\frac{x+y}{5} = \frac{y}{2}$$

or after cross-multiplying: $2x + 2y = 5y$ which is the same as: $2x = 3y$.

(b) How fast is the jogger's shadow growing when the jogger has gone 6 meters past the lamppost?

Solution: The goal is to find $\frac{dy}{dt}$, since y is the shadow length. Start with the simplest version of the answer from part (a): $2x = 3y$ and take the derivative of both sides with respect to t:

$$2\left(\frac{dx}{dt}\right) = 3\left(\frac{dy}{dt}\right).$$

The runner is moving away from the lamppost at 3 m/s so that is $\frac{dx}{dt}$. Substituting, we can solve for $\frac{dy}{dt}$. Therefore the shadow length y is increasing at a rate of $\frac{dy}{dt} = 2$ meters per second.

4. (9 points) Find the following antiderivatives.

(a) $\int 6x^2 + 2x + 1 \, dx$.

Solution: $2x^3 + x^2 + x + C$

(b) $\int \sqrt{x} \, dx$.

Solution: $\frac{2}{3}x^{3/2} + C$

(c) $\int 2x \cos(x^2) \, dx$.

Solution: (u-Substitution) Let $u = x^2$, then $du = 2x \, dx$ and $dx = \dfrac{du}{2x}$.

$$\int 2x \cos(x^2) \, dx = \int 2x \cos u \, \frac{du}{2x} = \sin u + C = \sin(x^2) + C.$$

5. (8 points) Let $f(x) = x(6-x)$.

(a) Find the area under the curve $y = x(6-x)$ from $x = 0$ to $x = 3$.

Solution:

$$\int_0^3 x(6-x)\,dx = \int_0^3 6x - x^2\,dx = \left[3x^2 - \tfrac{x^3}{3}\right]_0^3 = 27 - 9 = 18.$$

(b) Find average value of $f(x) = x(6-x)$ on the interval $[0,3]$.

Solution:

$$\frac{1}{3-0}\int_0^3 x(6-x)\,dx = \tfrac{1}{3}(18) = 6.$$

6. (6 points) Find the value of c that makes the following function continuous.

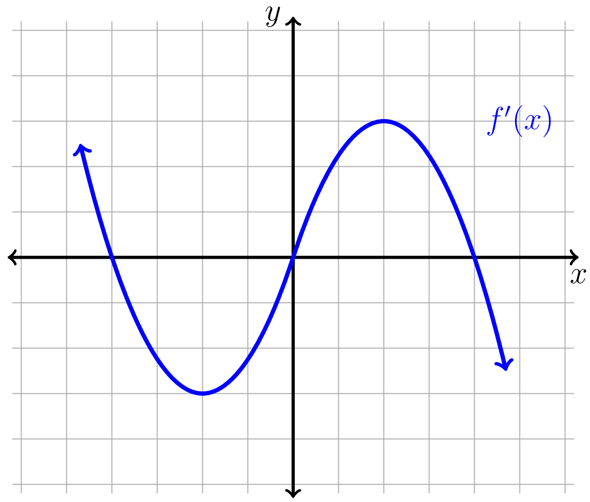
$$f(x) = \begin{cases} cx - 2 & \text{if } x > 3 \\ 1 - x^2 & \text{if } x \leq 3 \end{cases}$$

Solution: We just need the two pieces to touch when $x = 3$. At $x = 3$, the left piece is $1 - x^2 = -8$. The right piece is $cx - 2 = 3c - 2$. So we solve

$$3c - 2 = -8$$

to find $c = \frac{-6}{3} = -2$.

7. (8 points) The graph below shows the **derivative** $f'(x)$ of another function $f(x)$.



(a) Find the intervals where the original function $f(x)$ (which is not shown) is increasing.

> **Solution:** The original function is increasing when $f'(x)$ is positive. That looks like the intervals $(0, 4)$ and $(-\infty, -4)$. It is decreasing when $f'(x)$ is negative, i.e., $(-4, 0)$ and $(4, \infty)$.

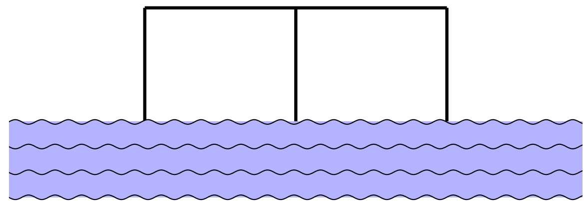
(b) Find the x-values of all local maxes and local mins for $f(x)$.

> **Solution:** Local extrema happen at critical points where the derivative is zero. Those are $x = -4, x = 0$, and $x = 4$. $x = -4$ and $x = 4$ are both local maxes and $x = 0$ is a local minimum.

(c) Use the graph of $f'(x)$ to find the intervals of concavity for $f(x)$.

> **Solution:** To find concavity, we need the derivative of $f'(x)$. Really, we just need to 2nd order critical points, which are when the derivative of $f'(x)$ is zero. Those appear to be $x = -2$ and $x = +2$ from the graph above. In the interval $(-2, 2)$, $f'(x)$ is increasing so $f''(x) > 0$ and $f(x)$ is concave up. In both intervals $(-\infty, -2)$ and $(2, \infty)$, $f'(x)$ is decreasing so $f''(x) < 0$ and $f(x)$ is concave down.

8. (8 points) A farmer has 2400 ft. of fence and wants to fence off a rectangular field that borders a straight river. He also wants to use fence to divide the field into two halves as shown below. No fence is needed along the side of the river.



(a) Write down a formula for the area of the field. If you use variables, clearly indicate what each letter represents.

Solution: Let y be the length of the three vertical fences in the picture, and let x be the length of the long horizontal size. Then area is xy.

(b) Find the dimensions of the field with maximum possible area given the amount of fence available.

Solution: The constraint is the amount of fence: $3y + x = 2400$, so $x = 2400 - 3y$. Therefore the area is

$$A(y) = (2400 - 3y)y = 2400y - 3y^2.$$

To find the max area, set the derivative to zero:

$$A'(y) = 2400 - 6y = 0 \Rightarrow y = 400$$

Then solve for $x = 2400 - 3(400) = 1200$. Finally, we just need to double check that our answer really is the maximum. The easiest way is to use the 2nd derivative test.

$$A''(y) = -6$$

which is always concave down, so our critical point must be a max.

9. (6 points) Use linear approximation or differentials to approximate $\sqrt[3]{8.24}$.

Solution: Use the linear approximation formula

$$f(x) \approx f(a) + f'(a)(x - a).$$

Here $f(x) = x^{1/3}$ and $a = 8$ because that is an easy number to plug into the formula that is close to the actual input value of $x = 8.24$. The derivative is $f'(x) = \frac{1}{3}x^{-2/3}$ and so $f'(a) = \frac{1}{3}(8^{-2/3}) = \frac{1}{6}$. Also $f(a) = \sqrt[3]{8} = 2$. Putting it all together,

$$\sqrt[3]{8.24} \approx 2 + \tfrac{1}{12}(0.24) = 2.02.$$

10. (6 points) Use the *definition of derivative* to find the derivative of $f(x) = x^2 + 4$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 4 - x^2 - 4}{h} =$$

$$= \lim_{h \to 0} \frac{x^2 - 2xh + h^2 + 4 - x^2 - 4}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x.$$

11. (6 points) A particle moves along the x-axis. The velocity of a particle is given by the formula $v(t) = 16t - 10$.

(a) What is the acceleration of the particle as a function of time?

Solution: Acceleration is the derivative of velocity.

$$a(t) = v'(t) = 16$$

(b) What is the position of the particle as a function of time, if the particle is at $x = 5$ when $t = 1$?

Solution: Position is the antiderivative of velocity:

$$x(t) = \int v(t)\, dt = 8t^2 - 10t + C.$$

To find C, use the initial condition $x(1) = 5$:

$$x(1) = 8 - 10 + C = C - 2 = 5$$

so $C = 7$ and $x(t) = 8t^2 - 10t + 7$.

12. (4 points) If $h'(t)$ represents the rate of growth of a tree in feet per year, then what does

$$\int_3^5 h'(t)\,dt$$

represent about the tree?

Solution: $\int_3^5 h'(t)\,dt$ is the total growth of the tree from year 3 until year 5.

13. (8 points) Consider the function $f(x) = \dfrac{2x^2+6}{x^2+2x-3}$.

(a) Does $f(x)$ have any vertical asymptotes? If so, where?

Solution: You get a vertical asymptote when you have a nonzero number divided by 0. That happens in the expression above when the bottom $x^2+2x-3 = (x+3)(x-1) = 0$. So when $x = 1$ and $x = -3$.

(b) Does $f(x)$ have any horizontal asymptotes? Where?

Solution: To find the horizontal asympytotes, calculate the limit of the function as x goes to $+\infty$ and $-\infty$. The easiest way to do that is to factor out the highest power of x:

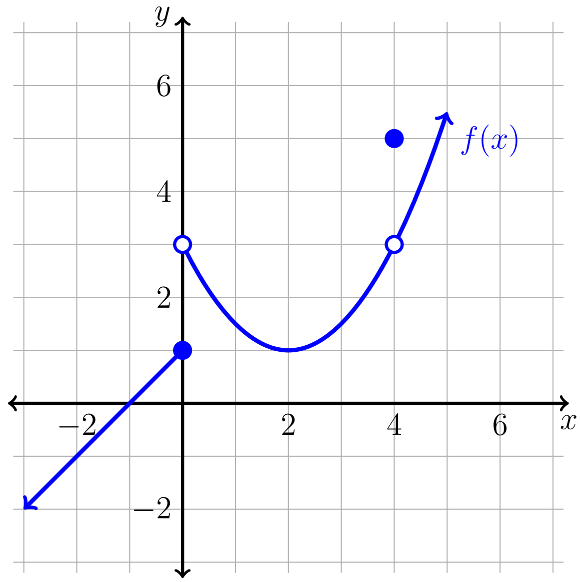
$$\lim_{x\to\infty} \frac{2x^2+6}{x^2+2x-6} = \lim_{x\to\infty} \frac{x^2(2+6/x^2)}{x^2(1+2/x-6/x^2)} = 2.$$

The limit as $x \to -\infty$ is the same, so there is only one horizontal asymptote at $y = 2$.

(c) Is $f(x)$ a continuous function everywhere? Why or why not?

Solution: No, the function is undefined at the vertical asymptotes $x = 1$ and $x = -3$. So it is not continuous at those two points.

14. (6 points) Use the graph below to find the indicated limits.



(a) $\lim_{x \to 0} f(x)$.

Solution: The limit from the left $\lim_{x \to 0^-} f(x) = 1$ and the limit from the right $\lim_{x \to 0^+} f(x) = 3$. Since the two sides don't agree, the limit does not exist.

(b) $\lim_{x \to 4} f(x)$.

Solution: The limit from the left $\lim_{x \to 4^-} f(x) = 3$ and the limit from the right $\lim_{x \to 4^+} f(x) = 3$. Since the two sides do agree, the limit is 3. It does not matter what the actual value is at $x = 4$.

Formula Sheet

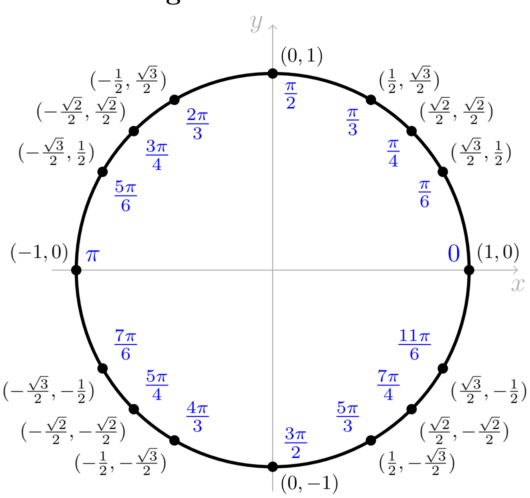
Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Common Angles



Trigonometry Ratios

- $\tan x = \dfrac{\sin x}{\cos x} \qquad \cot x = \dfrac{\cos x}{\sin x}$
- $\sec x = \dfrac{1}{\cos x} \qquad \csc x = \dfrac{1}{\sin x}$

Angle Addition Identities

- $\cos(a + b) = \cos a \cos b - \sin a \sin b$
- $\sin(a + b) = \sin a \cos b + \sin b \cos a$

Trigonometry Limits

- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Definition of Derivative

- $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$, or
- $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Selected Derivatives

- $\dfrac{d}{dx} \tan x = \sec^2 x$
- $\dfrac{d}{dx} \sec x = \sec x \tan x$
- $\dfrac{d}{dx} \cot x = -\csc^2 x$
- $\dfrac{d}{dx} \csc x = -\csc x \cot x$

Linear Approximation

- $f(x) \approx f(a) + f'(a)(x - a)$

Error and Relative Error

- $dy \approx$ the error in y
- $\dfrac{dy}{y} \approx$ the relative (percent) error in y

Newton's Method

- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Summation Formulas

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Riemann Sum

- $A \approx \displaystyle\sum_{n=1}^{N} f(x_n)\Delta x$ where
- $\Delta x = \dfrac{b - a}{N}$ and
- $x_n = a + n\Delta x$

Average Value of a Function

- $\dfrac{1}{b - a} \displaystyle\int_a^b f(x)\, dx$