Fixed Point Iteration **Math 342**

We will prove the following theorem.

> **Theorem.** Suppose that $f \in C^2[a,b]$ has a fixed point $p \in (a,b)$. If $|f'(p)| < 1$, then for any x_0 sufficiently close to p, there is a positive constant $C < 1$ such that the recursive sequence $x_{n+1} = f(x_n)$ has
> $$|x_{n+1} - p| \le C\,|x_n - p| \text{ for all } n \in \mathbb{N}.$$

1. Find the 1st degree Taylor approximation for $f(x)$ centered at p including the remainder term.

2. Let $M = \max_{a \le z \le b} |f''(z)|$. Use the triangle inequality to find an upper bound for $|x_{n+1} - p|$, assuming that $a \le x_n \le b$.

3. How small does $|x_n - p|$ need to be in order to guarantee that there is a positive constant $C < 1$ such that
$$\frac{|x_{n+1} - p|}{|x_n - p|} \le C?$$