The Legendre polynomials are not the only important family of orthogonal functions. The functions $\sin(nx)$ and $\cos(nx)$ with $n \in \mathbb{N}$ are a very important family of orthogonal functions in the inner-product space $L^2[-\pi, \pi]$. Combining continuous least squares regression with these functions, we get Fourier series.

Definition. For a function $f \in L^2[-\pi, \pi]$, the **Fourier series** for f is

$$f(x) = \frac{a_0}{2\pi} + \sum_{n=1}^{\infty} \left(a_n \frac{\cos(nx)}{\pi} + b_n \frac{\sin(nx)}{\pi} \right)$$

where

$$a_n = \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \text{ and } b_n = \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx.$$

1. Use Desmos to find the Fourier series approximation (up to $N = 3$) for the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

   on the interval $[-\pi, \pi]$. You can use Desmos to calculate the integrals and find the coefficients.

2. Calculate the coefficients of the Fourier series for $f(x) = x$ by hand. Then use Desmos to graph the Fourier series up to $N = 10$. What is the largest value of x such that the Fourier approximation is equal to $f(x)$ exactly?

3. How can you tell, without calculating anything, that $\cos(kx)$ and $\sin(mx)$ are orthogonal functions in $L^2[-\pi,\pi]$ when k and m are any integers?

4. A useful fact about Fourier series is that for any function f in $L^2[-\pi,\pi]$,

$$\|f(x)\|^2 = \frac{a_0^2}{2\pi} + \sum_{n=1}^{\infty} \frac{a_n^2}{\pi} + \sum_{n=1}^{\infty} \frac{b_n^2}{\pi}.$$

We can use this fact to solve the famous **Basel problem**: finding the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(a) Use the definition to calculate $\|f(x)\|^2$ when $f(x) = x$.

(b) Find the Fourier coefficients a_n and b_n for $f(x) = x$ for every $n \in \mathbb{N}$ (see problem 2).

(c) Use the formula for $\|f(x)\|^2$ in terms of the Fourier coefficients to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.