Runge-Kutta Method **Math 342**

The fourth order Runge-Kutta method (RK4) is commonly used to numerically approximate the solution of an initial value problem

$$\frac{dy}{dt} = f(t, y) \quad \text{with } t \in [a, b] \text{ and initial condition } y(a) = y_0.$$

The formula for updating the y-values in RK4 is

$$y_{i+1} = y_i + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + h/2, y_i + hk_1/2), \\ k_3 &= f(t_i + h/2, y_i + hk_2/2), \\ k_4 &= f(t_i + h, y_i + hk_3). \end{aligned}$$

1. Write a function in Python to implement RK4.

2. Use your RK4 function to approximate the solution to the IVP

$$\frac{dy}{dt} = \frac{y}{t} - \left(\frac{y}{t}\right)^2$$

   on the interval $[1, 2]$ with initial condition $y(1) = 1$ and $h = 0.1$. What y-value do you get for the right endpoint (when $t = 2$)?

3. The exact solution to this IVP is $y(t) = \dfrac{t}{1 + \ln t}$. Find the absolute error in the RK4 approximation of $y(2)$ and compare it with the absolute error in the Euler's method approximation (both with $h = 0.1$).

4. Use RK4 with $h = 0.01$ to approximate the solution of the differential equation

$$\frac{dy}{dt} = \frac{\cos t}{y^2 + 1}, \qquad y(0) = 1$$

on the interval $[0, 4\pi]$. Graph your solution.

5. Solve this IVP by hand. You don't need to find an explicit solution, but you should use the initial condition to solve for the constant C.

6. How could you (numerically) compute the value of $y(\pi/3)$ without using a numerical method for differential equations (like Euler's method or a Runge-Kutta)?