The following problems are similar to ones you might see on the midterm exam.

1. Use the method of divided differences to find the Newton basis interpolating polynomial for the points $(0, 0)$, $(1, 1)$, and $(4, 2)$.

Solution: The table of divided differences is

x	y	DD1	DD2
0	0		
		1	
1	1		$-\frac{1}{6}$
		$\frac{1}{3}$	
4	2		

So the interpolating polynomial is $p_2(x) = x - \frac{1}{6}x(x-1)$ in the Newton basis.

2. What is the interpolating polynomial above written in terms of the Lagrange basis polynomials?

Solution: The Lagrange basis polynomials are

$$L_0(x) = \frac{1}{4}(x-1)(x-4) \qquad L_1(x) = -\frac{1}{3}x(x-4) \qquad L_2(x) = \frac{1}{12}x(x-1)$$

so the interpolating polynomial is $p_2(x) = -\frac{1}{3}x(x-4) + \frac{1}{6}x(x-1)$ in the Lagrange basis.

3. Write down the Vandermonde matrix system for these same points $(0, 0)$, $(1, 1)$, $(4, 2)$ to find the coefficients of the interpolating polynomial in the standard basis. You don't need to solve the Vandermonde matrix system.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

4. The second degree interpolating polynomial for the function $f(x) = 1/x$ on the interval $[1, 3]$ with three equally spaced nodes ($x_0 = 1$, $x_1 = 2$, $x_2 = 3$) is

$$p_2(x) = \tfrac{1}{6}x^2 - x + \tfrac{11}{6}.$$

The error formula for interpolating polynomials with equally spaced nodes is

$$|f(x) - p_n(x)| \leq \frac{h^{n+1}}{4(n+1)} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$$

where $h = \frac{b-a}{n}$. Use this formula to find an upper bound on the error in using the polynomial p_2 to approximate $f(x) = 1/x$ on the interval $[1, 3]$.

Solution: In this example $n = 2$ and $h = 1$. The third derivative of f is $f^{(3)}(x) = -6x^{-4}$. The largest possible absolute value for this function would occur at the left endpoint when $x = 1$, so the error is bounded by:

$$\frac{h^{n+1}}{4(n+1)} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)| \leq \frac{1}{12} 6(1^{-4}) = \frac{1}{2}.$$

5. The formula for Simpson's method (not composite) on an interval is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a) + 4f(m) + f(b)\right) - \frac{h^5}{90} f^{(4)}(\xi)$$

where $h = \frac{b-a}{2}$ and $m = \frac{a+b}{2}$ and ξ is some value between a and b. Use this rule to approximate area under the function $y = e^x$ on the interval $[0, 2]$ and estimate the error in the approximation.

Solution: In this example, $a = 0$, $b = 2$, $m = 1$ and $h = 1$. So the approximate area is $\frac{1}{3}(1 + 4e + e^2) = 6.421$. The error in this approximation is $-\frac{h^5}{90} f^{(4)}(\xi) = -\frac{1}{90} e^{\xi}$. Since e^{ξ} is at most e^2 on the interval, the absolute value of the error is at most $\frac{e^2}{90} = 0.0821$.

6. To estimate the area under a curve using Gaussian quadrature you need to convert the function to an equivalent integral on the interval $[-1, 1]$. Then you can use Gaussian quadrature with any number of nodes. The formula for Gaussian quadrature with $n = 3$ nodes is

$$\int_{-1}^{1} f(x)\,dx \approx \tfrac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \tfrac{8}{9} f(0) + \tfrac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right).$$

Find an integral on $[-1, 1]$ that is equal to $\int_0^2 e^x\,dx$ and then use Gaussian quadrature to estimate the value of the integral.

Solution: Since both $[0, 2]$ and $[-1, 1]$ have the same length, you just need to translate the function to the left by one. You can do this by replacing the function $f(x) = e^x$ with a function $g(x) = e^{x+1}$. Then

$$\int_0^2 f(x)\,dx = \int_{-1}^{1} g(x)\,dx.$$

Using the function g in the formula for Gaussian quadrature I got an approximate area of

$$\tfrac{5}{9} g\left(-\sqrt{\tfrac{3}{5}}\right) + \tfrac{8}{9} g(0) + \tfrac{5}{9} g\left(\sqrt{\tfrac{3}{5}}\right) = \tfrac{5}{9} e^{1-\sqrt{3/5}} + \tfrac{8}{9} e + \tfrac{5}{9} e^{1+\sqrt{3/5}} = 6.389.$$

7. Use the centered difference quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

to approximate the derivative of $\tan x$ at $x = \pi/3$ with $h = 10^{-5}$. What is the relative error in your approximation?

Solution: The approximation is

$$\frac{\tan(\frac{\pi}{3} + 10^{-5}) - \tan(\frac{\pi}{3} - 10^{-5})}{2 \cdot 10^{-5}} = 4.0000000013584724.$$

The exact value of the derivative is

$$\sec^2(\pi/3) = 4.$$

So the relative error is

$$\frac{|4.0000000013584724 - 4|}{4} = 3.396 \times 10^{-10}.$$

8. The normal equation to find the coefficients of a (discrete) least square regression model is

$$X^T X b = X^T y.$$

Suppose you want the best fit linear function $\hat{y} = b_0 + b_1 x$ to approximate the points $(-2, 3)$, $(0, 2)$, $(2, 0)$.

(a) What is the matrix X and the vector y in the normal equation above?

Solution:

$$X = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

(b) Compute $X^T X$ and $X^T y$.

Solution:

$$X^T X = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \quad \text{and} \quad X^T y = \begin{bmatrix} 5 \\ -6 \end{bmatrix}.$$

(c) Solve the normal equations to find the slope and y-intercept of the regression line $\hat{y} = b_0 + b_1 x$.

Solution: The slope is $b_1 = -3/4$ and the y-intercept is $b_0 = 5/3$.

9. The Legendre polynomials are a family of orthogonal functions on the interval $[-1, 1]$. The first three Legendre polynomials are

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1).$$

Using the Legendre functions as a basis, find the best fit (continuous least squares) 2nd degree polynomial approximation of the function $\cos x$ on the interval $[-1, 1]$. You can use the following integrals to help

$$\int_{-1}^{1} P_0(x)\cos x\, dx = 1.683 \qquad \int_{-1}^{1} P_1(x)\cos x\, dx = 0$$

and

$$\int_{-1}^{1} P_2(x)\cos x\, dx = -0.124 \qquad \int_{-1}^{1} P_2(x)^2\ dx = 0.4.$$

Solution: The least squares solution is

$$\frac{\langle P_0, \cos x\rangle}{\langle P_0, P_0\rangle} P_0(x) + \frac{\langle P_1, \cos x\rangle}{\langle P_1, P_1\rangle} P_1(x) + \frac{\langle P_2, \cos x\rangle}{\langle P_2, P_2\rangle} P_2(x)$$

Four of those inner-products were given as integrals above. The only two left to calculate are:

$$\langle P_0, P_0\rangle = \int_{-1}^{1} 1\, dx = 2$$

and

$$\langle P_1, P_1\rangle = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$$

So the final answer is

$$\frac{1.683}{2} P_0(x) - \frac{0.124}{0.4} P_2(x) = 0.8415 - 0.31\left(\tfrac{1}{2}(3x^2 - 1)\right).$$