Math 342 Workshop - Inner Products   **Name:** _____

The **inner product** of two (column) vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ is $\mathbf{x}^T\mathbf{y}$. Two vectors are **orthogonal** if their inner product is zero. The 2-norm of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$.

1. Explain why $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$ for every pair of vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$. Hint: You can assume that $x_1, \ldots, x_n$ are the entries of $\mathbf{x}$ and $y_1, \ldots, y_n$ are the entries of $\mathbf{y}$.

2. The **law of cosines** says that if a, b, c are the lengths of the sides of any triangle and θ is the angle between the sides of lengths a and b, then $c^2 = a^2 + b^2 - 2ab\cos\theta$. Use the law of cosines to prove that for any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$,

$$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \cos\theta$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. Hint: use the inner-product to find $\|\mathbf{x} - \mathbf{y}\|^2$.
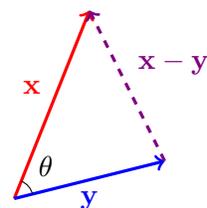
The **orthogonal complement** of a set V in $\mathbb{R}^n$ is the set

$$V^\perp = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w}^T\mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}.$$

That is, $V^\perp$ is the set of all vectors that are orthogonal to *every* vector in V.

3. Suppose that V is the set containing just one vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Describe the set $V^\perp$ algebraically. What can you say about the entries x_1, x_2, x_3 of a vector $\mathbf{x}$ in $V^\perp$?

   (b) Describe the set $V^\perp$ geometrically. What is the shape of the set $V^\perp$?

4. Let W be the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Describe the set $W^{\perp}$ both algebraically and geometrically.

5. Let I denote the identity matrix, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ with $\|\mathbf{x}\| = 1$. Let $P = I - \mathbf{x}\mathbf{x}^T$.

   (a) Show that $P^2 = P$. Hint: FOIL $(I - \mathbf{x}\mathbf{x}^T)(I - \mathbf{x}\mathbf{x}^T)$.

   (b) Show that for every $\mathbf{y}$ in $\mathbb{R}^n$, $P\mathbf{y}$ is orthogonal to $\mathbf{x}$. Hint: Simplify $\mathbf{x}^T P\mathbf{y}$.