Formula Sheet

Quadratic formula

$$ax^2 + bx + c = 0 \text{ when } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Absolute error

$$|\text{approximate value} - \text{exact value}|$$

Relative error

$$\frac{|\text{absolute error}|}{|\text{exact value}|}$$

Taylor's theorem

$f(x) = P_n(x) + R_n(x)$ where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

for some ξ between x and c.

Important Maclaurin series

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
- $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$
- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Secant method

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Newton's method for systems

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{J}(\mathbf{x}_n)^{-1}\mathbf{F}(\mathbf{x}_n)$$

Condition number

$$\kappa(A) = \|A\| \, \|A^{-1}\|$$

Orthogonal projection onto line spanned by y

$$\text{Proj}_{\mathbf{y}}(\mathbf{x}) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}\mathbf{y}$$

Gram-Schmidt algorithm

$$\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \sum_{j=0}^{k} \frac{\langle \mathbf{x}_{k+1}, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle}\mathbf{v}_j$$

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(k\pi x) + b_k \sin(k\pi x))$$

where

$$a_k = \int_{-1}^{1} f(x)\cos(k\pi x)\,dx, \qquad b_k = \int_{-1}^{1} f(x)\sin(k\pi x)\,dx$$

Lagrange polynomials

$$L_k(x) = \frac{\prod_{j=0,\ j\neq k}^{n}(x - x_j)}{\prod_{j=0,\ j\neq k}^{n}(x_k - x_j)}$$

Interpolation error

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$$

for some ξ between x and the x_i's.

Composite midpoint method error

$$|\text{Error}| \leq \max_{a \leq \xi \leq b} \frac{|f^{(2)}(\xi)|(b-a)^3}{24n^2}$$

Composite Simpson's method error

$$|\text{Error}| \leq \max_{a \leq \xi \leq b} \frac{|f^{(4)}(\xi)|(b-a)^5}{2880n^4}$$

Euler's method

For $y' = f(t, y)$ and $h = (b-a)/n$,

$$y_{k+1} = y_k + hf(t_k, y_k)$$
$$t_{k+1} = t_k + h$$