Numerical Analysis - Math 342 **Final Exam Review**

The final exam will be on ***Thursday, April 30 at 9:00am****. The following problems are similar to ones you might see on the final exam.*

1. Let $f(x) = x^3 + 2x - 8$.
   (a) Apply Newton's method twice to $f(x)$ starting with $x_0 = 1$. What are x_1 and x_2?

   (b) If you were to continue applying Newton's method in this case, the sequence x_k would converge to a number x_∞. What is $f(x_\infty)$?

   (c) If you start with $x_0 = 0$ and $x_1 = 1$, then what is x_2 according to the secant method?

2. Consider the four points $(-2, 0), (0, 6), (1, 18), (3, 30)$.
   (a) Write the interpolating polynomial using Lagrange polynomials.

   (b) Make a table of divided differences for these four points.

   (c) Write the interpolating polynomial using the Newton basis.

3. Use the Simpson's method error formula

$$|\text{Error}| \le \max_{a\le\xi\le b} \frac{|f^{(4)}(\xi)(b-a)^5}{2880n^4}$$

to find the minimum number of subintervals needed to guarantee that the you can approximate the integral $\int_0^2 e^{-x^2}\,dx$ with an absolute error less than 10^{-8}. Hint: if $f(x) = e^{-x^2}$, then the maximum value of $|f^{(4)}(\xi)|$ is 12 on the interval $[0, 2]$.

4. Use Euler's method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = x - y^2 \text{ on } [0,3] \text{ with initial condition } y(0) = 1.$$

Use $n = 3$ and fill in the table below.

x	0			
y				

5. Compute the LU-decomposition of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

6. The monic Legendre polynomials are an orthogonal family of polynomials on the interval $[-1, 1]$. The first three monic Legendre polynomials are $L_0(x) = 1$, $L_1(x) = x$, and $L_2(x) = x^2 - \frac{1}{3}$. The $L^2[-1, 1]$ norms of these polynomials are given by $\|L_0\|^2 = 2$, $\|L_1\|^2 = \frac{2}{3}$, and $\|L_2\|^2 = \frac{8}{45}$.
   (a) What is the $L^2[-1, 1]$ norm of the function $f(x) = x^4$?

   (b) Find the orthogonal projection of the function $f(x) = x^4$ onto the span of the first three monic Legendre polynomials.

7. The polynomial $p_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$ is the 4th degree Maclaurin polynomial for e^x. Suppose you use $p_4(-2)$ to estimate the value of e^{-2}. What is the worst case error in that approximation, according to Taylor's remainder theorem?

8. The function $f(x) = e^{-0.1x}$ has a fixed point near $x_0 = 1$. Suppose you calculate $x_1 = f(x_0)$, then $x_2 = f(x_1)$, and then continue calculating $x_{n+1} = f(x_n)$ recursively. (i) How can you tell that this recursive process will converge? (ii) What is the fixed point, accurate to four decimal places?

9. Find two orthogonal vectors that have the same span as $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ -10 \\ 2 \end{bmatrix}$.

10. Recall the composite trapezoid rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)).$$

If you could calculate the function f exactly every time, then the error in the composite trapezoid rule would be:

$$|\text{Error}| \leq \max_{a \leq \xi \leq b} |f^{(2)}(\xi)| \frac{(b-a)^3}{12n^2}.$$

In reality, you typically cannot calculate the function f exactly because of rounding errors.

(a) Assume that every time the function f is computed there is an (absolute) error of up to $\delta \approx 10^{-16}$. What is the worst case total rounding error in the formula above?

(b) What happens to the rounding error in the limit as n gets very large? Does it keep getting smaller or does it start getting bigger eventually? Explain.

11. How many floating point operations are needed to multiply two n-by-n matrices?