

The following problems are similar to ones you might see on the midterm exam.

1. Use Newton's method to write down an iterative formula for finding the root of  $f(x) = x^3 - a$  for any constant  $a$ . If you start with the initial guess  $x_0 = \frac{1}{3}a$ , then what is  $x_1$ ?
2. The root of  $x^3 - 2$  is  $\sqrt[3]{2}$ , which is located in the interval  $[1, 2]$ . If we use the bisection method to find this root, starting with the endpoints  $a = 1$  and  $b = 2$ , then what is the worst case error in our estimate for the root after 10 steps?
3. Find values for the constants  $M$  and  $L$  such that  $|f''(x)| \leq M$  and  $|f'(x)| \geq L$  when  $f(x) = x^3 - 2$  on the interval  $[1, 2]$ .
4. Based on your constants from the previous problem, and the Newton's method error formula

$$|x_{n+1} - r| \leq \left( \frac{M}{2L} \right) |x_n - r|^2,$$

how close to the root  $r$  would a guess  $x_n$  in  $[1, 2]$  need to be in order to guarantee that the next iterate  $x_{n+1}$  is definitely closer to  $r$ ?

5. Use the triangle inequality to find an upper bound  $M$  for  $|f'(x)|$  when  $f(x) = \sin 2x + \cos 3x$ .
6. Find the fixed points of the function  $f(x) = \frac{8}{3x-2}$ .
7. What is the derivative of the function  $f(x) = \frac{8}{3x-2}$  at each fixed point? Based on the derivative, determine whether each fixed point is attracting or repelling (or not enough information).

8. Let  $A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 7 & 21 \\ 1 & 11 & 1 \end{pmatrix}$ .

(a) Find the LU-decomposition of  $A$ .

(b) What is the rank of  $A$ ? Is  $A$  invertible?

(c) Compute  $\|A\|_\infty$ .

(d) Use the LU-decomposition to solve  $Ax = \begin{pmatrix} 2 \\ 11 \\ -1 \end{pmatrix}$ .

9. Suppose that  $x = 1.234 \times 10^{-3}$  and  $y = 1.225 \times 10^{-3}$  each have four significant digits. How many significant digits are there in each of the following numbers?

(a)  $x + y$ .

(b)  $x - y$ .

(c)  $xy$ .

(d)  $x/y$ .

10. Let  $f(x) = \frac{e^x - 1}{x}$ .

(a) Find a Maclaurin polynomial for  $f$  by replacing  $e^x$  by its 3rd degree Maclaurin polynomial.

(b) Find a formula for the error in the previous approximation using the Taylor remainder formula. What is an upper bound for the error on  $[-1, 1]$ ?

11. If you use the secant method to find the root of  $y = 2^x - 5$  starting with  $x_0 = 1$  and  $x_1 = 2$ , what is  $x_2$ ?

12. Express the following system of equations as a vector equation  $\mathbf{F}(\mathbf{x}) = \mathbf{0}$  and find the Jacobian  $\mathbf{J}(\mathbf{x})$ .

$$\begin{aligned} x^2 - 4y^2 &= 1 \\ x^2 + xy &= 1 \end{aligned}$$

13. Draw a rough sketch of a cobweb diagram for the function  $f(x) = -\frac{1}{2}x + 5$  starting with  $x_0 = 1$ .

