Numerical Analysis - Math 342 **Midterm 2 Review**

The following problems are similar to ones you might see on the midterm exam.

1. How many floating point operations does it take to compute $y = a + bx + cx^2$ if a, b, c and x are all floating point numbers?

2. It is possible to re-write the polynomial above as $y = a + x(b + cx)$. How many floating point operations would it take to compute this alternative formula for the polynomial?

3. Use the method of divided differences to find the Newton basis interpolating polynomial for the points $(0, 0)$, $(1, 1)$, and $(4, 2)$.

4. What is the interpolating polynomial above written in terms of the Lagrange basis polynomials?

5. Write down the Vandermonde matrix system for these same points $(0, 0)$, $(1, 1)$, $(4, 2)$ to find the coefficients of the interpolating polynomial in the standard basis. You don't need to solve the Vandermonde matrix system.

6. The second degree interpolating polynomial for the function $f(x) = 1/x$ on the interval $[1, 3]$ with three equally spaced nodes ($x_0 = 1$, $x_1 = 2$, $x_2 = 3$) is

$$p_2(x) = \tfrac{1}{6}x^2 - x + \tfrac{11}{6}.$$

The error formula for interpolating polynomials with equally spaced nodes is

$$|f(x) - p_n(x)| \leq \frac{h^{n+1}}{4(n+1)} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$$

where $h = \frac{b-a}{n}$. Use this formula to find an upper bound on the error in using the polynomial p_2 to approximate $f(x) = 1/x$ on the interval $[1, 3]$.

7. Let $\mathbf{x} \in \mathbb{R}^n$ be any vector with $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = 1$. Let I denote the n-by-n identity matrix.

   (a) Simplify the following expression as much as possible.

$$(I - 2\mathbf{x}\mathbf{x}^T)(I - 2\mathbf{x}\mathbf{x}^T).$$

   (b) What does the answer to part (a) imply about the columns of the matrix $I - 2\mathbf{x}\mathbf{x}^T$?

8. The normal equation to find the coefficients of a (discrete) least square regression model is

$$X^T X b = X^T y.$$

Suppose you want the best fit linear function $\hat{y} = b_0 + b_1 x$ to approximate the points $(-2, 3)$, $(0, 2)$, $(2, 0)$.

(a) What is the matrix X and the vector y in the normal equation above?

(b) Compute $X^T X$ and $X^T y$.

(c) Solve the normal equations to find the coefficients of the regression line $\hat{y} = b_0 + b_1 x$.

9. The Legendre polynomials are a family of orthogonal functions on the interval $[-1, 1]$. The first three Legendre polynomials are

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = x^2 - \tfrac{1}{3}.$$

Using the Legendre polynomials as a basis, find the best fit (continuous least squares) 2nd degree polynomial approximation of the function $\cos x$ on the interval $[-1, 1]$. Express your answer with coefficients that are accurate to 4 significant digits. You can use the following integrals to help

$$\int_{-1}^{1} P_0(x) \cos x \, dx = 1.683 \qquad \int_{-1}^{1} P_1(x) \cos x \, dx = 0$$

and

$$\int_{-1}^{1} P_2(x) \cos x \, dx = -0.08271 \qquad \int_{-1}^{1} P_2(x)^2 \, dx = \frac{8}{45}.$$

10. Apply the Gram-Schmidt algorithm to the vectors to get an orthogonal basis for $\mathbb{R}^3$.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}.$$

11. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and let $\mathbf{y} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ -1 \end{bmatrix}$.

    (a) Find the orthogonal projection of $\mathbf{y}$ onto the span of $\mathbf{x}$.

    (b) Apply the Gram-Schmidt algorithm to $\{\mathbf{x}, \mathbf{y}\}$ to get two orthogonal vectors with the same span.

12. Find the inner product of $f(x) = 3x^2 - 1$ and $g(x) = x^2$ on the interval $[-1, 1]$.

13. Explain how you can tell that the functions $f(x) = x^4 + 5$ and $g(x) = \sin x$ are orthogonal in $L^2[-1, 1]$ without actually integrating.