

The following problems are similar to ones you might see on the midterm exam.

1. How many floating point operations does it take to compute $y = a + bx + cx^2$ if a, b, c and x are all floating point numbers?

Solution: There are two additions, and three multiplications (including one to compute x^2) for a total of 5 flops.

2. It is possible to re-write the polynomial above as $y = a + x(b + cx)$. How many floating point operations would it take to compute this alternative formula for the polynomial?

Solution: Now there are two additions and two multiplications for a total of 4 flops.

3. Use the method of divided differences to find the Newton basis interpolating polynomial for the points $(0, 0)$, $(1, 1)$, and $(4, 2)$.

Solution: The table of divided differences is

x	y	DD1	DD2
0	0		
1	1	1	
4	2	$\frac{1}{3}$	$-\frac{1}{6}$

So the interpolating polynomial is $p_2(x) = x - \frac{1}{6}x(x - 1)$ in the Newton basis.

4. What is the interpolating polynomial above written in terms of the Lagrange basis polynomials?

Solution: The Lagrange basis polynomials are

$$L_0(x) = \frac{1}{4}(x - 1)(x - 4) \quad L_1(x) = -\frac{1}{3}x(x - 4) \quad L_2(x) = \frac{1}{12}x(x - 1)$$

so the interpolating polynomial is $p_2(x) = -\frac{1}{3}x(x - 4) + \frac{1}{6}x(x - 1)$ in the Lagrange basis.

5. Write down the Vandermonde matrix system for these same points $(0, 0)$, $(1, 1)$, $(4, 2)$ to find the coefficients of the interpolating polynomial in the standard basis. You don't need to solve the Vandermonde matrix system.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

6. The second degree interpolating polynomial for the function $f(x) = 1/x$ on the interval $[1, 3]$ with three equally spaced nodes ($x_0 = 1$, $x_1 = 2$, $x_2 = 3$) is

$$p_2(x) = \frac{1}{6}x^2 - x + \frac{11}{6}.$$

The error formula for interpolating polynomials with equally spaced nodes is

$$|f(x) - p_n(x)| \leq \frac{h^{n+1}}{4(n+1)} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$$

where $h = \frac{b-a}{n}$. Use this formula to find an upper bound on the error in using the polynomial p_2 to approximate $f(x) = 1/x$ on the interval $[1, 3]$.

Solution: In this example $n = 2$ and $h = 1$. The third derivative of f is $f^{(3)}(x) = -6x^{-4}$. The largest possible absolute value for this function would occur at the left endpoint when $x = 1$, so the error is bounded by:

$$\frac{h^{n+1}}{4(n+1)} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)| \leq \frac{1}{12} 6(1^{-4}) = \frac{1}{2}.$$

7. Let $\mathbf{x} \in \mathbb{R}^n$ be any vector with $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}} = 1$. Let I denote the n -by- n identity matrix.

(a) Simplify the following expression as much as possible.

$$(I - 2\mathbf{x}\mathbf{x}^T)(I - 2\mathbf{x}\mathbf{x}^T).$$

Solution: First FOIL

$$(I - 2\mathbf{x}\mathbf{x}^T)(I - 2\mathbf{x}\mathbf{x}^T) = I - 2\mathbf{x}\mathbf{x}^T - 2\mathbf{x}\mathbf{x}^T + 4\mathbf{x}\mathbf{x}^T\mathbf{x}\mathbf{x}^T$$

Then cancel $\mathbf{x}^T\mathbf{x}$ in the last term, since that is just 1.

$$= I - 2\mathbf{x}\mathbf{x}^T - 2\mathbf{x}\mathbf{x}^T + 4\mathbf{x}\mathbf{x}^T$$

Then collect the like terms and everything cancels except for I . So the answer is I .

(b) What does the answer to part (a) imply about the columns of the matrix $I - 2\mathbf{x}\mathbf{x}^T$?

Solution: It means that the columns of $I - 2\mathbf{x}\mathbf{x}^T$ are an orthonormal set.

8. The normal equation to find the coefficients of a (discrete) least square regression model is

$$X^T X b = X^T y.$$

Suppose you want the best fit linear function $\hat{y} = b_0 + b_1 x$ to approximate the points $(-2, 3)$, $(0, 2)$, $(2, 0)$.

- (a) What is the matrix X and the vector y in the normal equation above?

Solution:

$$X = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

- (b) Compute $X^T X$ and $X^T y$.

Solution:

$$X^T X = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \quad \text{and} \quad X^T y = \begin{bmatrix} 5 \\ -6 \end{bmatrix}.$$

- (c) Solve the normal equations to find the coefficients of the regression line $\hat{y} = b_0 + b_1 x$.

Solution: The slope is $b_1 = -3/4$ and the y-intercept is $b_0 = 5/3$.

9. The Legendre polynomials are a family of orthogonal functions on the interval $[-1, 1]$. The first three Legendre polynomials are

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = x^2 - \frac{1}{3}.$$

Using the Legendre polynomials as a basis, find the best fit (continuous least squares) 2nd degree polynomial approximation of the function $\cos x$ on the interval $[-1, 1]$. Express your answer with coefficients that are accurate to 4 significant digits. You can use the following integrals to help

$$\int_{-1}^1 P_0(x) \cos x \, dx = 1.683 \quad \int_{-1}^1 P_1(x) \cos x \, dx = 0$$

and

$$\int_{-1}^1 P_2(x) \cos x \, dx = -0.08271 \quad \int_{-1}^1 P_2(x)^2 \, dx = \frac{8}{45}.$$

Solution: The least squares solution is

$$\frac{\langle P_0, \cos x \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle P_1, \cos x \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle P_2, \cos x \rangle}{\langle P_2, P_2 \rangle} P_2(x)$$

Four of those inner-products were given as integrals above. The only two left to calculate are:

$$\langle P_0, P_0 \rangle = \int_{-1}^1 1 \, dx = 2$$

and

$$\langle P_1, P_1 \rangle = \int_{-1}^1 x^2 dx = \frac{2}{3}.$$

So the final answer is

$$\frac{1.683}{2}P_0(x) - \frac{0.08271}{8/45}P_2(x) = 0.8415 - 0.4652(x^2 - \frac{1}{3}).$$

10. Apply the Gram-Schmidt algorithm to the vectors to get an orthogonal basis for \mathbb{R}^3 .

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}.$$

Solution:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{v}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{v}_1, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{-9}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}.$$

11. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and let $\mathbf{y} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ -1 \end{bmatrix}$.

(a) Find the orthogonal projection of \mathbf{y} onto the span of \mathbf{x} .

Solution:

$$\text{Proj}_{\mathbf{x}}(\mathbf{y}) = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} \mathbf{x} = \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

(b) Apply the Gram-Schmidt algorithm to $\{\mathbf{x}, \mathbf{y}\}$ to get two orthogonal vectors with the same span.

Solution:

$$\mathbf{v}_1 = \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{v}_2 = \mathbf{y} - \text{Proj}_{\mathbf{x}}(\mathbf{y}) = \begin{bmatrix} 5 \\ 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}.$$

12. Find the inner product of $f(x) = 3x^2 - 1$ and $g(x) = x^2$ on the interval $[-1, 1]$.

Solution:

$$\langle f, g \rangle = \int_{-1}^1 (3x^2 - 1)x^2 dx = \int_{-1}^1 (3x^4 - x^2) dx = 2 \left[\frac{3x^5}{5} - \frac{x^3}{3} \right]_0^1 = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}.$$

13. Explain how you can tell that the functions $f(x) = x^4 + 5$ and $g(x) = \sin x$ are orthogonal in $L^2[-1, 1]$ without actually integrating.

Solution: Because $f(x)$ is an even function and $g(x)$ is an odd function, the product $f(x)g(x)$ is an odd function. So its integral from -1 to 1 will be zero.